B.Sc-II

Equations of first order & first degree

Homogeneous Equations

A function $f(x,y)$ is homogeneous of degree n if it can be expressed in the form $x^n \phi(\frac{y}{x})$.

If M and N are homogeneous functions of x and y of the same degree n, then the equation $Mdx + Ndy = 0$ is said to be a homogeneous equation of degree n. In this case we may write

$$M = x^n \phi(y/x) \;,\; N = x^n \psi(y/x)$$

on the substitution of $y = vx$, so that v may be considered a new dependent variable, the equation $Mdx + Ndy = 0$ becomes

$$x^n \phi(v)\, dx + x^n \psi(v) \{x dv + v dx\} = 0$$

$$\Rightarrow x^n \{[\phi(v) + v\psi(v)]\, dx + x\psi(v) dv\} = 0$$

$$\Rightarrow \frac{dv}{F(v)} + \frac{dx}{x} = 0 \quad (\because x^n \neq 0)$$

$$\text{Where } F(v) = \frac{v\psi(v) + \phi(v)}{\psi(v)} = v + \frac{\phi(v)}{\psi(v)}$$

Thus, the variables are separated & the solution is

$$\int \frac{dv}{F(v)} = \log \frac{c}{x}$$

The primitive will be given by the substitution of $\frac{y}{x}$ for v after the integration has been performed.

Example① Solve $2xy\frac{dy}{dx} = y^2 - x^2$.

Solution: → Here the equation is homogenous, for

$$\frac{dy}{dx} = \frac{y^2-x^2}{2xy} = \frac{\left(\frac{y}{x}\right)^2 - 1}{2\frac{y}{x}}$$

Substituting $y = vx$, we get

$$v + x\frac{dv}{dx} = \frac{v^2-1}{2v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^2-1}{2v} - v = \frac{v^2-1-2v^2}{2v}$$

$$\Rightarrow \quad \frac{2v\,dv}{1+v^2} + \frac{dx}{x} = 0$$

Integrating, we have

$$\log(1+v^2) + \log x = \log c$$

$$\Rightarrow \quad x(v^2+1) = c$$

∴ The primitive is

$$x\left(\frac{y^2}{x^2}+1\right) = c \quad \text{or,} \quad cx = y^2 + x^2$$

Ans.

Example② Solve $x^2 dy + y(x+y)dx = 0$

Solution: → $\because \frac{dy}{dx} = -\frac{y(x+y)}{x^2}$ is Homogenous.

put $y = vx$, where v is variable. Then

$$v + x\frac{dv}{dx} = -\frac{vx(x+vx)}{x^2} = -v(1+v)$$

$$\Rightarrow x\frac{dv}{dx} = -v(v+2)$$

$$\Rightarrow \frac{dx}{x} + \frac{dv}{v(v+2)} = 0$$

Integrating, we get

$$\cancel{\log x}\int\frac{dx}{x} + \frac{1}{2}\int\left[\frac{1}{v} - \frac{1}{v+2}\right]dv = 0$$

$$\Rightarrow \log x + \frac{1}{2}\log\frac{v}{v+2} = \log c$$

$$\Rightarrow \log x^2 + \log\frac{y}{y+2x} = \log c$$

$$\Rightarrow \log\frac{x^2y}{y+2x} = \log c \Rightarrow \frac{x^2y}{2x+y} = c$$

Therefor, the solution is: $x^2y = c(2x+y)$

<u>Ans.</u>

<u>Example③:</u> → Solve $(y^4 - 2x^3y)dx + (x^4 - 2xy^3)dy = 0$

<u>Solution:</u>→

Here,

$$M = y^4 - 2x^3y = x^4\left\{\left(\frac{y}{x}\right)^4 - 2\left(\frac{y}{x}\right)\right\}$$

$$\& \; N = x^4 - 2xy^3 = x^4\left\{1 - 2\left(\frac{y}{x}\right)^3\right\}$$

$\therefore$ $Mdx + Ndy = 0$ becomes

$$\left\{\left(\frac{y}{x}\right)^4 - 2\left(\frac{y}{x}\right)\right\}dx + \left\{1 - 2\left(\frac{y}{x}\right)^3\right\}dy = 0 \quad (\because x^4 \neq 0)$$

The substitution $y = vx$ gives

$$(v^4 - 2v)\,dx + (1-2v^3)(x\,dv + v\,dx) = 0$$

$$\Rightarrow (v^4 - 2v + v - 2v^4)\,dx + x(1-2v^3)\,dv = 0$$

$$\Rightarrow \frac{dx}{x} = \frac{1-2v^3}{v+v^4}\,dv = \left(\frac{1}{v} - \frac{3v^2}{1+v^3}\right)dv$$

Integrating, we have

$$\log x = \log v - \log(1+v^3) + \log c$$

$$\Rightarrow \quad x(1+v^3) = cv$$

i.e; $x^3 + y^3 = cxy \qquad (\because v = \frac{y}{x})$

<u>Ans.</u>

<u>Example ④</u> Solve. $(x^2-y^2)\,dy = xy\,dx$

<u>Example ⑤</u>: → Solve

$$2xy\,dx + (y^2-x^2)\,dy = 0$$

<u>Example ⑥</u>: → Solve

$$(x+y)\frac{dy}{dx} = y-x$$